

NAVIGATING CURVED SPACES

NAVIGATING CURVED SPACES:

HANDS-ON EXPLORATIONS IN
HYPERBOLIC AND ELLIPTIC GEOMETRY

MICHAEL MCDANIEL, PHD



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*Navigating Curved Spaces:
Hands-On Explorations in Hyperbolic and Elliptic Geometry*

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Navigating the curved spaces

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Navigating the curved spaces

Introduction

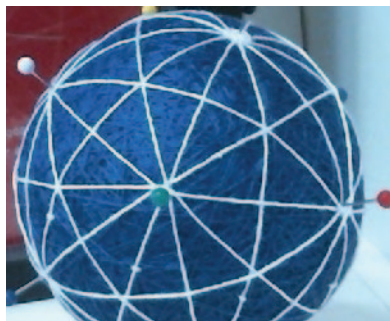
Geometry has two curved spaces which can be modeled on sheets of paper.

Negative curves away from itself.



Hyperbolic space.

Positive curves toward itself.



Elliptic space.

Angle sum of triangle less than 180° .

Angle sum of triangle more than 180° .

In this book, the reader explores curved spaces as creator and adventurer. Speed gets priority. Instead of proving every geometric property, we lay out ideas to use which have been proved elsewhere and build on those. The author refers to the book *Geometry by Construction* as a source of proofs and ideas but access to that book is not required. That book and this one, however, have been written to play nice together.

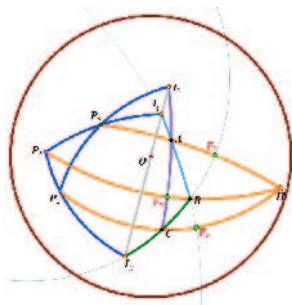
The curved spaces have significance in art, architecture, medicine, navigation and other human-sized activities. Outside what we see, curvature plays a part in the subatomic and the celestial.

When we have learned enough geometry for an informed discussion, we will entertain some oddball conjectures about what we see and do not see. The ideas are not totally crazy because some ideas match empirical evidence. For us, the ideas come first because math people have often functioned in that manner. Examples: the differential equations for planetary motion predicted the location of Neptune, Einstein needed a non-Euclidean space-time for his ideas, and navigating the oceans during the day used spherical trigonometry. Math people had the required knowledge ahead of the need.

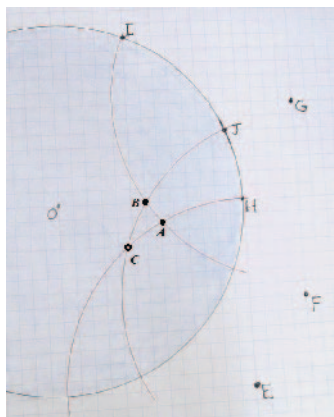
The reader should not skip to the end because this stuff is baffling enough when assimilated in small doses. To tackle oddball conjectures, we ask the reader to experience bite-size craziness. Anybody who wants their craziness in huge, unmanageable chunks is asking for trouble. The reader may have tackled

enough big math problems for a lifetime. A math person, however, carries around a tough mathematical question on purpose. Their unconscious mind chips away at it and sends potential advances upstairs to the waking mind. When the math person gets stuck in an airport, out comes a pencil and a sheet of paper. In the chapters ahead, the reader gets to study examples of some of the types of worlds a math person recreates on a sheet of paper every time they get the chance to work.

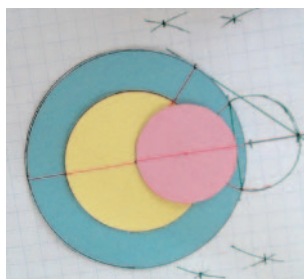
How to work in these places.



Dynamic geometry.



Construction by hand.



Collage.



Photos of objects.

We will use four methods of illustration in navigating the curved spaces. Well, more than illustration; we will construct the objects we study. We can use a dynamic geometrical system, as in the first picture, which gives us precise control and lots of nice colors. We will use the ancient method of drawing compass and straightedge, often on graph paper. The graph paper and a T-square device will help us make fast right angles when we need them. Creating on paper gives us the human connection to our work, much like working through a calculus problem by hand. There's just some gut-level give-and-take which

we acquire when we engage in the physical act of creation. The drawing might not be perfect, but we *really* get the picture because we made it ourselves.

A few times, we get to use collage because the ability to manipulate the physical pieces will pay off with a combination of vision and placement crucial to understanding a subtle twist in our situation. We will use photos of useful objects in our journeys. The author's apologies here because holding the physical objects would give a deeper experience. The photos will have to do. At least they will be photos of familiar things rather than fantastical objects from a curved space outside our experience (until this book.)

For the reader who does not plan on actually performing the constructions, considering the structures remains key to navigating the non-Euclidean spaces. Physicists and chemists have achieved some great results doing thought experiments; so such a reader should think. Merely skimming the text and glancing at the pictures will not stimulate the deeper ideas required for savoring the unusual ideas ahead. Imagine the satisfaction in achieving some fluency in hyperbolic geometry and then getting some of the Questions right.

The reader might also like to skip some of the calculations and equations in the text. Again, the reader's satisfaction might benefit from examining the mathematical thinking behind the ideas. A math person should want to read some calculations and read constructions. If there is no grade riding on this book's contents, fine. Skip an equation's simplification. Naughty, naughty!

Some familiarity with the mathematical imagination could benefit anyone. All the artists, writers and dreamers exploring this book: *Steal These Ideas*. Darn few creators use non-Euclidean geometry in their productions. Read the book with selfishness.

The World's Fastest Trip to Hyperbolic Space

What you need to know:

How to draw a circle with a compass.

Get your sheet of graph paper with some other paper underneath so your compass point can get a grip.

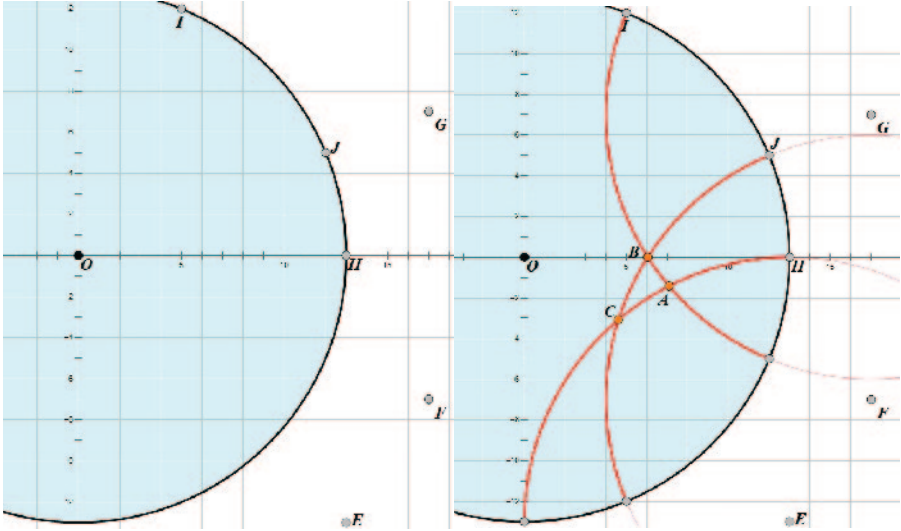
- ✓ Stick the point of your compass precisely where you want it, usually an intersection of lines on the graph paper.

- ✓ Before you draw, pivot the compass on its point to make sure the compass will draw where you intend. All lined up?

- ✓ DRAW THAT CIRCLE ONCE, clean.

For this journey, we use a pencil, one sheet of graph paper, a drawing compass, and a ruler. Set the compass to the size of 13 units on the graph paper. Mark a point in the middle of the paper and label it O . Now place the point of the compass at O and draw a circle of radius 13 units on the page. Keep the compass that size.

Welcome to hyperbolic space!



Let's make some hyperbolic lines. Mark the points $H(13, 0)$, $I(5, 12)$, $J(12, 5)$, $E(13, -13)$, $F(17, -7)$, and $G(17, 7)$.

Then use the points E, F, G as centers and draw circles of radius 13 from each center.

The circles meet inside circle O and form a shark fin whose vertices we will call A, B, C .

The reader familiar with 5, 12, 13 has recognized a Pythagorean triple. The sides of a right triangle may have integer lengths. Such trios, like 3, 4, 5 and 5, 12, 13 satisfy the Pythagorean theorem.

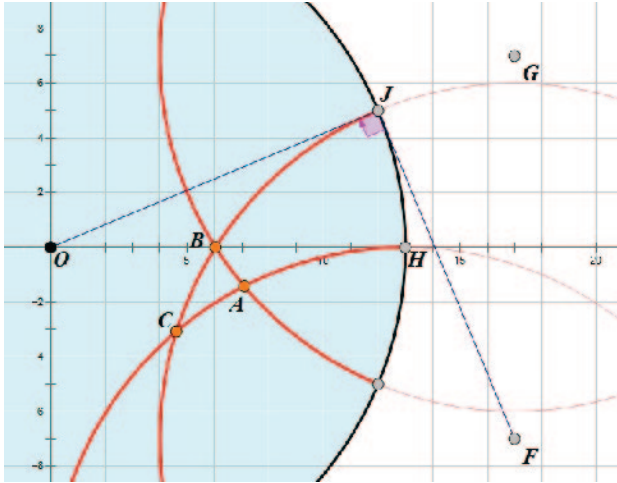
$$3^2 + 4^2 = 9 + 16 = 25 = 5^2 \text{ and } 5^2 + 12^2 = 13^2$$

This ancient idea enables us to construct hyperbolic lines in hyperbolic space with ease. Henri Poincaré developed the hyperbolic disk around 1900, about 2200 years after the Greek mathematician Euclid. This disk version of two dimensional hyperbolic space gets called the Poincaré disk or the hyperbolic disk interchangeably. Euclidean geometry was the only geometry for many long years until non-Euclidean geometries, like hyperbolic, were discovered.

The corner of circles

The first circle, O , is the boundary of the Poincaré disk. All the points inside the disk are hyperbolic points, which appear light blue. Fortunately, these points are exactly the same as Euclidean points. Each point is pure location. A point has no size. The points on the circle itself are not hyperbolic points; in fact the points on the circle and outside the circle are just plain Euclidean points and are not part of hyperbolic space at all.

The three circles we have drawn are orthogonal to circle O . Look at point J on circle O . The two circles which meet at J have radii ending at J and those radii are perpendicular. That is how a pair of circles form a corner.



Which brings us to a strange situation. A hyperbolic line in the disk is the arc of a circle inside the disk and the circle must be orthogonal to the disk. So we have some hyperbolic lines and these lines are also arcs of Euclidean circles. We will call hyperbolic lines with two letter names, just like Euclidean lines. Check this out: we will allow the letters to be the names of points on the boundary. So hyperbolic line $\overleftrightarrow{CJ}$ is the arc of circle F inside circle O and the totally oddball part is, the hyperbolic line $\overleftrightarrow{CJ}$ has a point in its name which is, technically, not a hyperbolic point, J . The most important requirement is that we always know what we are talking about. How annoying would it be to construct hyperbolic objects but be unable to apply obvious names? We will see that the points on the boundary circle O have various important roles in hyperbolic geometry. So we might as well get used to allowing their use from the beginning.

Mathematics does not function on the honor system. "Trust me, this is a hyperbolic line," does not hold. We cannot simply apply the adjective "hyperbolic" to something and that thing is now hyperbolic. There is no such thing as a hyperbolic V-8 combustion engine. The status of being hyperbolic has to be established. We have constructed corners of circles along circle O : now we will see why some things are hyperbolic.

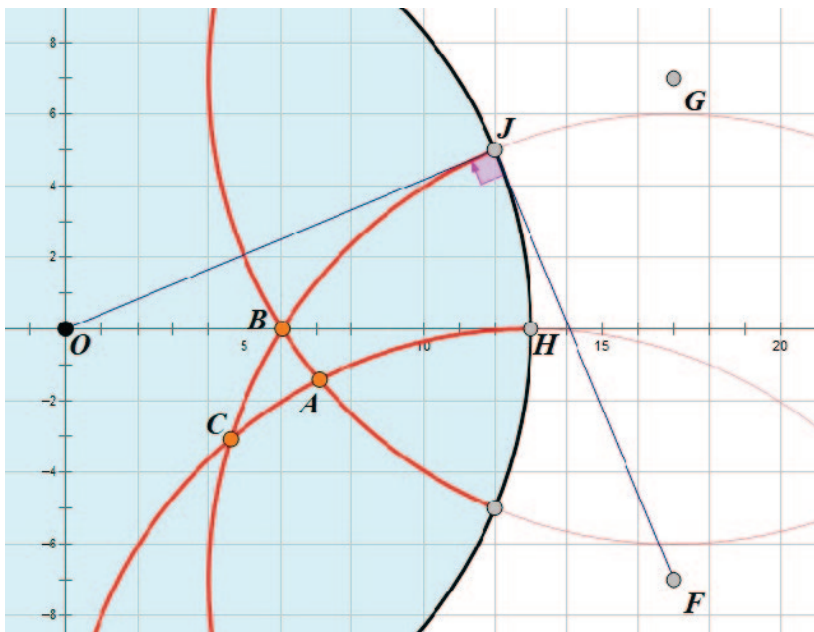
What makes this stuff hyperbolic?

A corner of circles where one circle is circle O means the other circle contains a hyperbolic line. When we want to construct a hyperbolic line which looks curved, we have to construct a corner of circles using circle O . That requires planning.

We can verify our instant hyperbolic lines with a little work. Establishing our claims makes hyperbolic space different from comic books and fantasies and essays where people just make stuff up. As we explore hyperbolic space, we explore a real thing with its own rules and properties, most of which are unknown to us as we start. Unlike Oz or Tatooine, the facts of hyperbolic space are their own facts, not reliant on an author to lay down the structure. Hyperbolic space has no Dungeon Master.

Wittgenstein wrote that the world is that which is the case. Hyperbolic space is that which is the case. If we point to a circle intersecting circle O and say, "a corner of circles happens here," we could be lying. We need to establish the intersection is an actual corner. What we do next previews all our curved space explorations: we establish the facts to our satisfaction. An established fact is the case, which makes our new fact part of hyperbolic space. In other words, being hyperbolic is earned. To be a hyperbolic line, the object must be a diameter or the arc of a circle orthogonal to circle O . To be a hyperbolic triangle, the sides must lie on hyperbolic lines.

Verifying the circles are orthogonal means that we check for perpendicular radii. Consider the circle E centered at $(13, -13)$ with radius 13. That circle has to be tangent to the horizontal line through point O at point H . In fact, the two radii from E to the points $(13, 0)$ and $(0, -13)$ form a square with the horizontal and vertical radii from O . So we have a genuine corner of circles at point H . Then the arc of the circle E inside circle O is the hyperbolic line $\overleftrightarrow{AC}$. We used Euclidean facts to establish that hyperbolic line $\overleftrightarrow{AC}$ is the case.

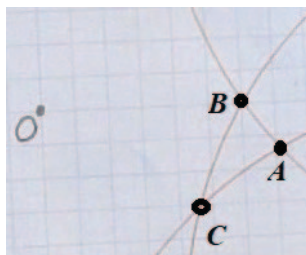


The circle F meets circle O at the point $J(12,5)$. We can establish perpendicular radii meeting there using the slope formula from algebra. We also need the property that lines whose slopes multiply to -1 are perpendicular lines.

The slope of Euclidean line $\overleftrightarrow{FJ}$ is $\frac{-7-5}{17-12} = \frac{-12}{5}$; The slope of $\overleftrightarrow{JO}$ is $\frac{5-0}{12-0} = \frac{5}{12}$. The product of these two fractions is -1 , so the radii at J are perpendicular. We have another hyperbolic line $\overleftrightarrow{BC}$, an established fact.

Hyperbolic triangle ABC

Now we have three hyperbolic lines which intersect by pairs, forming a hyperbolic triangle ABC . We had this triangle from the start but we had to take a second to establish that its sides were on hyperbolic lines.



The angles of our triangle look pinched compared to the angles of a Euclidean triangle. One of the weird properties of hyperbolic geometry is that the sum of

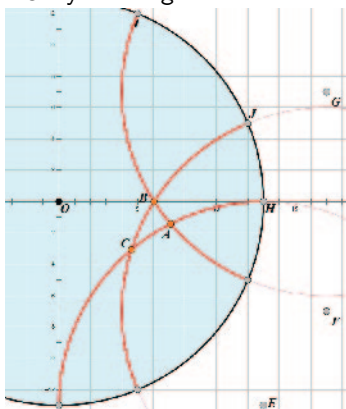
the angles of a hyperbolic triangle is not a fixed amount but is always less than the 180° for Euclidean triangles.

The Question at the end of this section asks for the creation of a triangle which is congruent to this triangle. We can have congruent hyperbolic triangles, two triangles which are the same size and same shape. We can use SSS, SAS and ASA to establish their congruence; *Geometry by Construction* has examples. However, we cannot have similar triangles in hyperbolic geometry. In fact, if two hyperbolic triangles have three pairs of congruent corresponding angles, the triangles are congruent! So hyperbolic geometry has a way of proving triangles congruent which Euclidean does not have: AAA.

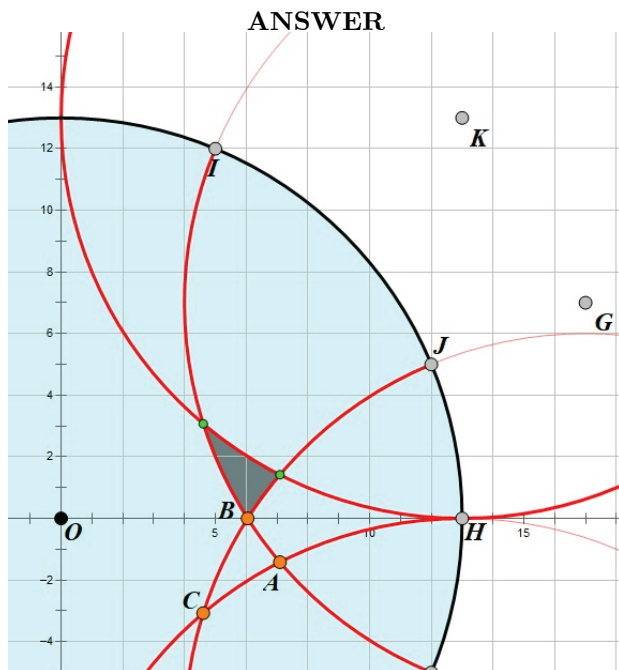
A Euclidean segment not lying on a Euclidean diameter of circle O is just a connected set of hyperbolic points. We may think of it as some kind of hyperbolic curve because the segment is not a hyperbolic segment. One of the hallmarks of playing in the hyperbolic disk is that we may use Euclidean properties to establish hyperbolic properties, and vice-versa. Sometimes we have enough hyperbolic facts to find out further hyperbolic facts without any Euclidean background. Sometimes we use combinations of ideas from both geometries. Such flexibility puts this sort of exploration in the AI-resistant category because how could a set of algorithms know when to apply Euclidean ideas and when to apply hyperbolic ideas?



Starting with our construction, construct a hyperbolic triangle congruent to triangle ABC by drawing one more circle of Euclidean radius 13.



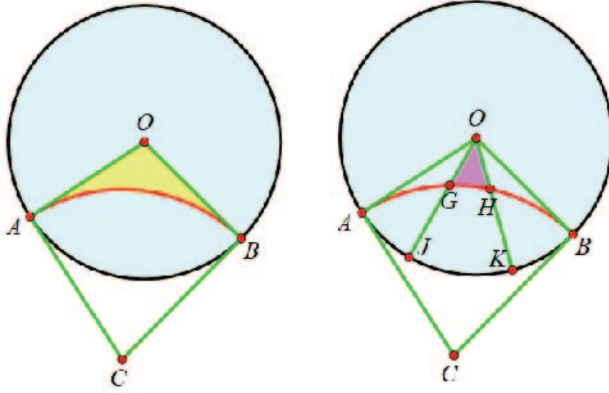
This Question is the first of many in this book. The reader should stop and think about the answer. The next page shows the solution with an explanation.



Point K (13, 13) plays the same role as point E for triangle ABC : the center of a Euclidean circle which contains a hyperbolic line. Point K is on the other side of the horizontal axis from point E . Circle K makes a new triangle which is the reflection of triangle ABC across undrawn line $\overleftrightarrow{OH}$.

The two triangles have to be congruent because by a reflection, one triangle would land exactly on the other. Point K is the reflection of point E across horizontal line $\overleftrightarrow{OH}$.

A teacher could use these Question and Answers to assess their students in a course studying non-Euclidean geometry. The Questions also point toward further questions. For instance, in this case, the pair of congruent triangles could be built again on the other side the vertical axis through circle O . The graph paper and same-size compass trick would have to deliver the same drawing in a new place.



Built for speed.

We can construct a quick hyperbolic triangle another way. Give yourself two radii in circle O and construct their tangents so they intersect at point C . Use C as the center of a circle through the points of tangency, A and B . The yellow-shaded triangle OAB in the first figure above is not a hyperbolic triangle because points A and B are not in hyperbolic space. We do, however, have a hyperbolic line, $\overleftrightarrow{AB}$. The shaded region is called a doubly-asymptotic triangle. We have two angles with measure 0° , $\angle OAB$ and $\angle OBA$.

The second figure finishes the job of constructing a fast hyperbolic $\triangle OGH$. We can draw any two radii inside $\angle AOB$ to get the intersections of hyperbolic lines which form the vertices of our hyperbolic triangle, shaded purple. The $\triangle OGH$ lies entirely in the disk, so the triangle is a hyperbolic triangle. Depending on our choice of $\overrightarrow{OJ}$ and $\overrightarrow{OK}$, the triangle could be isosceles or not. We could control the $\angle GOH$, if we wanted, and then we would know its angle size. Exploring what more could be designed into such a triangle might make for an interesting afternoon.

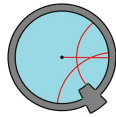
The reader might see that placing G and H symmetrically across the undrawn line $\overleftrightarrow{OC}$ would give an isosceles hyperbolic triangle OGH . The line of symmetry would be orthogonal to $\overleftrightarrow{GH}$. The properties of an isosceles triangle would trigger and we would have two congruent right hyperbolic triangles comprising triangle OGH . That's three modifiers on the word "triangles," an accomplishment in itself.

With further tinkering, the reader may discover the base angles of isosceles triangle OGH can be close to 0° when G is near A and close to 90° when G is near H . In all cases, the sum of the angles has to be less than 180° because triangle OGH is hyperbolic. At least the author considers such musings to be a good time.

Now we can construct our own hyperbolic triangles: a good start. Let's take a minute to combine Euclidean and hyperbolic ideas to make more hyperbolic lines and triangles. In the next question, we use the Euclidean distance formula, which is the Pythagorean theorem, to verify our hyperbolic structures.

From our first question, we could easily expand our triangle constructions to the other quadrants, but we would continue to work on a circle with Euclidean radius 13. In mathematics, we like to generalize an idea to extend our knowledge as far as that idea will stretch. Our quick trip into hyperbolic space using the Pythagorean triple (5, 12, 13) cannot be the only such trip. The only special property these three integers have is that they satisfy the Pythagorean theorem, which puts some points with integer coordinates on circle O .

So our second question seeks generalization.



Question

Given Pythagorean triple (a, b, c) ,
lay out all the details of another
quick trip into hyperbolic space,
constructing another triangle ABC .
Then prove the circles which need
to be orthogonal are indeed orthogonal.

Let's get organized. When we lay down what happened in our first hyperbolic trip, we will be able to see what should happen in the general case.

O	$(0, 0)$	$(\ , \)$
E	$(13, 13)$	$(\ , \)$
F	$(17, -7)$	$(\ , \)$
G	$(17, 7)$	$(\ , \)$
H	$(13, 0)$	$(\ , \)$
I	$(5, 12)$	$(\ , \)$
J	$(12, 5)$	$(\ , \)$

ANSWER

Based on the Pythagorean triple (a, b, c) such that $a < b < c$, we can guess the formulas for the coordinates from the example. We have to prove the corners of circles work no matter what.

O		$(0, 0)$	$(0, 0)$
E		$(13, -13)$	$(c, -c)$
F		$(17, -7)$	$(a + b, a - b)$
G		$(17, 7)$	$(a + b, b - a)$
H		$(0, 13)$	$(0, c)$
I		$(5, 12)$	(b, a)
J		$(12, 5)$	(a, b)

Let's take a moment to make sure a compass set to radius c will fit the same as our example. The distance from E to H is obviously c . The length

$$EH = \sqrt{(a + b - b)^2 + (a - b - a)^2} = \sqrt{a^2 + b^2} = c,$$

because we have a Pythagorean triple. The length

$$GJ = \sqrt{(a + b - a)^2 + (b - a - b)^2} = \sqrt{a^2 + b^2} = c.$$

Now we have a corner of circles to establish.

The slope formulas, in terms of a, b, c , should work because they worked with actual numbers which fit the Pythagorean theorem, just like the letters. Now all our points use the Pythagorean triple letters instead of actual numbers.

The slope of line $\overleftrightarrow{OI}$ is $\frac{b}{a}$.

The slope of line $\overleftrightarrow{GI}$ is $\frac{b - (b - a)}{a - (a + b)} = -\frac{a}{b}$.

The product of their slopes gives the required -1 . Any other corner of circles works just as well.

It's so nice to be able to make a hyperbolic space with some hyperbolic triangles quickly. To explore hyperbolic space, we certainly need hyperbolic things. Here's a bit of wisdom to help us through the rest of our hyperbolic ideas.

The position of a hyperbolic triangle in the hyperbolic disk is a property of that triangle! The Euclidean appearance of a hyperbolic object depends on its location. We will see hyperbolic circles with the same hyperbolic area which have different Euclidean sizes merely by being in different places in the hyperbolic disk.

A Euclidean triangle in the Euclidean plane can slide around and the triangle keeps its size. The same is true for a hyperbolic triangle in the disk. Yet its Euclidean appearance usually changes when the triangle moves. We have to think of a hyperbolic object as adjustable in its Euclidean look, not made of cardboard. Since we are beginners, our examples of this strange property will be hyperbolic triangles and circles.

Further considerations

The piper must be paid. We have indeed created two triangles in hyperbolic space with relatively little effort. Discerning further properties of triangle ABC takes a minute. Working through the Euclidean to get to the hyperbolic feels like translating a paragraph of hyperbolic into Euclidean and then trying to figure out what hyperbolic stuff was being said *in the hyperbolic geometry itself*. Hyperbolic fluency, the ability to think in the hyperbolic, is a goal for the next fifty pages. Perhaps achieving this will bring a sense of true accomplishment.

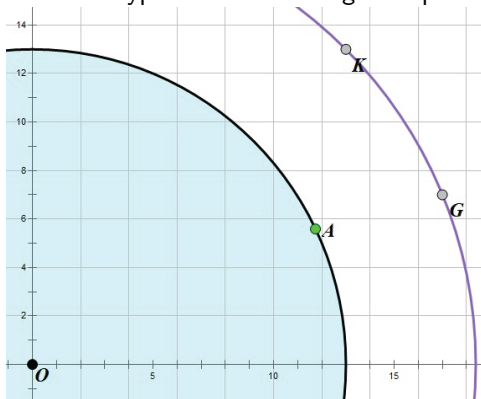
Thinking in hyperbolic would be a big deal because of the scope of this accomplishment. The disk model allows us to see the entire hyperbolic world. We do not get to see the entire Earth or even the entire Euclidean plane. If the reader agrees with the start of Wittgenstein's *Tractatus*, the ability to see the entire hyperbolic world is an opportunity outside our earthly experience. We cannot see all that is the case on Earth. Granted, our hyperbolic triangle ABC does not reveal the entirety of hyperbolic space to us. We do, however, have a model of an entire space based on a few ideas, which has appeared to us, incarnate on a sheet of graph paper.

We have already experienced the difference between merely saying something is so and establishing a fact about a hyperbolic object. The disk model awaits further exploration in this manner: establishing more true things which hold for the entire hyperbolic world. This could be more satisfying than finding a sort-of-true idea which only holds in Spring semester while the professor is in the room. The reader has opportunities ahead to discover new ideas and establish those ideas are, indeed, true. The new and true facts turn out to be intriguing in both their discovery and their verification.

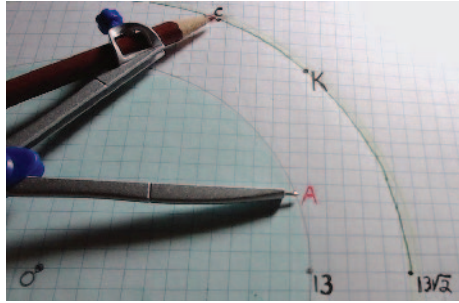


Question

Given concentric circles with a point on each as pictured, construct a hyperbolic line through the point A .



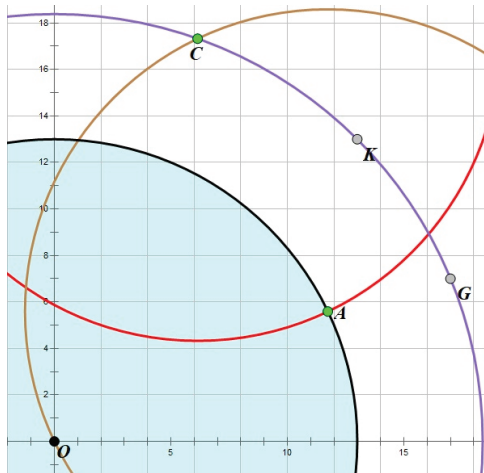
ANSWER



The vague question has a fast answer, but only an attentive reader could put together all the clues. The graph paper has point K at the meeting of two Euclidean (graph paper) lines tangent to the shaded circle O and these two tangents resemble some of the tangents used in the section. The undrawn segment $\overline{OK}$ would be a diagonal of a square we can see on the graph paper grid.

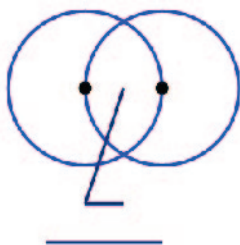
Now comes the flexible thinking of the attentive reader. Any point on the outer circle will serve as a center for a circle which will draw a hyperbolic line so long as we keep the compass the sized 13 because K was just the right distance from O . Now, to get the hyperbolic line to pass through point A , we have to find the place for the point of the compass. The pen will draw through point A , if we have chosen wisely.

Let's use point A to get what we want. That point C to the left of K in the picture came from putting the point of the compass at point A and drawing an arc which intersects the outer circle. (Does the reader see we could have found a second such location on the other side of point K ? Good.)



Point C is precisely the correct location to place the point of our compass and draw the red arc pictured: a hyperbolic line through A , drawn almost for free.

Cross-eyed nuisance



What you need to know

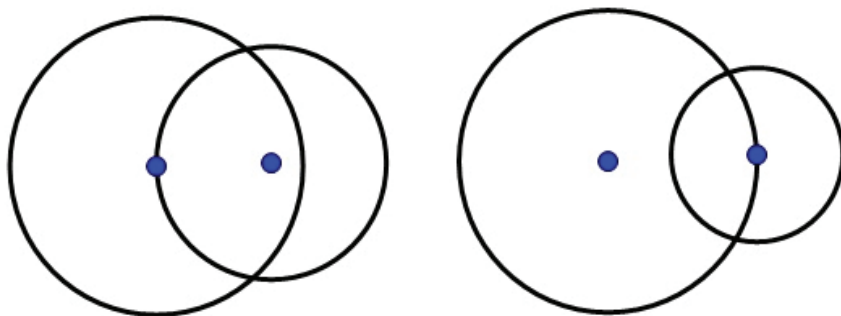
✓ We will need a Euclidean midpoint of a segment.

Set the compass bigger than half the length of the given segment $\overline{AB}$. Place the point of the compass on point A and make an arc above and below segment $\overline{AB}$. Do the same from point B , making sure the drawn arcs intersect. The ruler laid across the intersections will draw the perpendicular bisector of $\overline{AB}$, thus marking the midpoint of $\overline{AB}$.

✓ We need to be able to construct perpendicular Euclidean segments. The author likes the use of a T-square or any device which makes a corner. Just lay it down and draw that right angle.

For the moment, let's play in just plain Euclidean geometry.

In Euclidean geometry, if two circles go through each other circle's center, the two circles have to be the same size. There's a reason for this, staring us in the face in the figure above. Let's pretend for a minute we do not see the reason. What could happen if the two circles were different sizes? We immediately run into difficulties trying to get the circles to pass through each other's centers.



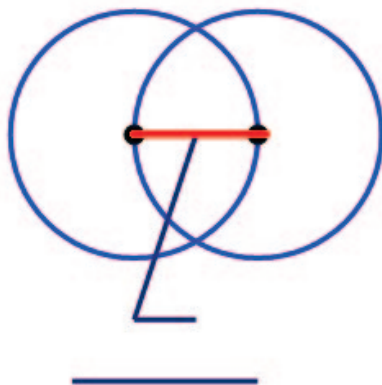
The two drawings above constitute an attempt to consider the situation with two circles of different size. We could argue that we have now considered the

smaller circle containing the larger circle's center and the larger circle containing the smaller circle's center. In each case, one circle misses the other's center. Would this argument be good enough for math people?

That's part of the fun, really. An appreciation for levels of sophistication happens in math. A person could counter the argument above by saying, let the larger circle in the first picture grow smaller until it goes through the other center. Could that still be the larger circle?

We'd like to shoot down this question with a reason. Let's do so.

If two circles pass through each other's centers, then they have one radius in common. Like the segment connecting the pupils in the first picture.

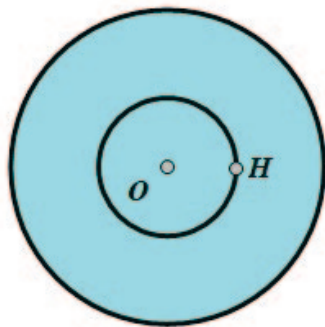


One radius in common means the two circles have the same size radius. The two circles are the same size. That iron-clad reason clinches the argument, ends the discussion, prevents anymore "what if..." scenarios.

In math, we always seek that sort of locked-down resolution to a question.

The hyperbolic situation looks different

Check out the hyperbolic version of this problem. Here's the given: the Poincaré disk with a hyperbolic circle centered at O . A point H has been marked on circle O . We desire to construct a circle through O with hyperbolic center H . Remember, the larger circle O is just the boundary of hyperbolic space. The circle we seek to construct has to fit in the space. We are allowed to construct Euclidean things outside the circle O , but hyperbolic things have to be fully inside.



The circle we seek would give us two circles with the radius $\overline{OH}$. These two circles would then have the same hyperbolic size. In order to construct this new circle, we need its Euclidean center, an unknown point E . Wait a second. The circle we seek has two centers?

Yes. The given circle has one center, point O , but the circle we seek to construct has a Euclidean center and a hyperbolic center. That's the deal because only one point can be both the Euclidean center and the hyperbolic center of a hyperbolic circle, point O . The Poincaré disk has center O . Any diameter of the disk is a hyperbolic line and a Euclidean segment. So a circle centered at O has a bunch of Euclidean segments as hyperbolic diameters and these all meet at O , making O the hyperbolic center because it is on every hyperbolic diameter.

Think of the Poincaré disk as a circular leaf of kale. The outer part is frilly, lots of wrinkled leaf space but the crinkles gradually flatten out as we look toward the center of the leaf. A little bit of area near the edge has more leaf material than a little bit of area toward the middle. Look down on the Poincaré disk with your eye directly above the point O and imagine more material as you look toward the boundary. There's room for a Euclidean circle to pass through O with both its hyperbolic center at H and its location entirely in the disk. So a hyperbolic circle of the same size hyperbolic radius will appear smaller than the given smaller circle O , to our Euclidean eyes.

The Euclidean center is the center we use to draw the hyperbolic circle and we do not have it. Not only can we construct this point, we can give an iron-clad reason for its being correct. The hyperbolic center is the hyperbolic point which is equidistant, in hyperbolic measure, to all the points on the hyperbolic circle. It is the place where all the hyperbolic diameters cross.

We know we can construct this hyperbolic center because of Chaim Goodman-Strauss's proof that Euclidean tools in the disk give us what hyperbolic tools would give us. Part of his proof was to show how, given a hyperbolic point as hyperbolic center and another hyperbolic point as a point on the circle, we can construct the Euclidean center of the desired circle. Which means our construction must exist. Since we're using a given circle centered at O , our construction